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Longitudinal Motion of an Elastic Bar

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SUMMARY
The exact equations of motion of an elastic bar are discussed, both in material and local coordinates. It is shown that for an ideal elastic material the former, but not the latter, are linear. An infinite number of conservation laws is shown to exist.

1. Introduction

In introductory courses in mathematical physics longitudinal waves in elastic bars or air columns and transverse waves on taut inextensible strings furnish instructive examples of methods used in more advanced topics. Usually the first treatment is restricted to linearised versions of these theories. In order to do this one has to gloss over rather off-handedly some critical points. In both theories there is, apart from the energy equation, a second conservation law quadratic in the variables. The interpretation of this law is one of the difficulties. In the elastic problem the question arises under which circumstances an elastic medium behaves like a linear field. In the string problem some difficulties exist in the derivation of the equation and the interpretation of the so called potential energy term. As far as we know the complete answers to these questions are not-readily available in ordinary text books. It might be of some interest therefore to describe how these difficulties can be circumvented by first setting up the exact theories, postponing linearisation to a later stage. The incentive to do this was a recent paper by Gilbert and Mollow [1] in which some aspects of the bar problem were treated. We start therefore with this subject. It is intended to treat the string problem in a second paper.

2. Coordinate Systems

The main defect of the elastic bar as a model for linear field theories like electrodynamics is the fact that elasticity (and hydrodynamics) is a convective field theory. Field quantities like strain and energy are carried by the moving matter. It is necessary therefore to distinguish carefully between local and material coordinates or, as it is called in hydrodynamics, Eulerian and Lagrangian coordinates. In one-dimensional problems material coordinates usually lead to the simplest equation (see e.g. Courant and Friedrichs [2]), we therefore adopt these as a starting point. An obvious choice for the material coordinate, which tells us with which material element we have to do, is m, the mass of the bar to the left of the element considered. An equivalent choice is s, the localisation of the element when the element $m=0$ of the unstrained bar is in the origin. Obviously:

$$m=\rho_0 s=\frac{s}{V_0} \tag{2.1}$$

where ρ_0 is the unstrained density, V_0 the unstrained specific volume (more correctly specific length).

The motion of the bar is completely specified when we know where each material element is at each instant of time. That is, we have to derive an equation for the function $x(m, t)$ or $x(s, t)$.

We observe that:

$$x_t = \left(\frac{\partial x}{\partial t}\right)_m = v \tag{2.2}$$

and

$$x_m = \left(\frac{\partial x}{\partial m}\right)_t = V \tag{2.3}$$

where v is the material velocity, V the specific volume. m and t have to be considered as independent coordinates:

$$m_t = 0\,. \tag{2.4}$$

This is the equation for the conservation of mass in the present coordinate system.

Finally we note that

$$x_{mt} = x_{tm}$$

and therefore:

$$V_t = v_m\,. \tag{2.5}$$

This is a purely kinematic condition. It simply states that the rate of stretching equals the mass-gradient of velocity.

3. Constitutive Equations

In order to set up the equation of motion we have to specify the relevant material properties of the bar. According to the theory of elasticity this is most easily done by assuming that the specific deformation energy E is known as a function of the specific volume V. The derivative of this function then is the local stress:

$$\sigma(V) = \frac{dE}{dV} = E'(V)\,. \tag{3.1}$$

Formally $-\sigma$ is equivalent to the pressure in the hydrodynamic analogue of our problem. Therefore the quantity:

$$H = E - \sigma V \tag{3.2}$$

is of the nature of a specific enthalpy.

The elastic modulus or Young's modulus $Y(V)$ is defined by:

$$Y = \sigma' = E''\,. \tag{3.3}$$

From (3.2) and (3.3) we deduce:

$$H' = -YV\,. \tag{3.4}$$

In the unstrained state we have $\sigma(V_0)=0$, $E(V_0)=H(V_0)$. The latter quantity can be taken as zero.

An ideal elastic medium is defined as a medium with a constant $Y=Y_0$. We have then, from (3.2) and (3.3):

$$\sigma = Y_0(V-V_0) \tag{3.5}$$

$$E = \tfrac{1}{2}Y_0(V-V_0)^2 \tag{3.6}$$

$$H = \tfrac{1}{2}Y_0(V_0^2-V^2)\,. \tag{3.7}$$

As E has the dimension of a velocity squared we can write (3.6) also in the form:

$$E = \tfrac{1}{2}c^2 \left(\frac{V}{V_0} - 1\right)^2 \tag{3.8}$$

where $c^2 = Y_0 V_0^2$. It will turn out that c is the small-amplitude speed of sound.

4. Equation of Motion and Conservation Laws

As the problem at hand is a purely mechanical one we assume that the Lagrangian density is the difference between specific kinetic and deformation energies. Using (2.3) we have:

$$L = \tfrac{1}{2}x_t^2 - E(x_m)\,. \tag{4.1}$$

Varying x we obtain the equation of motion:

$$x_{tt} = \{E'(x_m)\}_m\,. \tag{4.2}$$

With the aid of (2.2), (3.1) and (3.3) this can be written in the equivalent forms:

$$v_t = \sigma_m = YV_m\,. \tag{4.3}$$

We observe that (4.2) and the first equation (4.3) are in the form of a conservation equation. According to the well-known Noether theorem (see e.g. Hill [3]) this is a consequence of the fact that L contains only differentials of x and not this quantity itself. The same is true for t and m. Therefore there ought to be two other independent conservation equations. The equation corresponding to a variation of t is expected to be the energy equation. It turns out to be:

$$\{\tfrac{1}{2}x_t^2 + E(x_m)\}_t = (x_t E')_m$$

or

$$(\tfrac{1}{2}V^2 + E)_t = (v\sigma)_m \tag{4.4}$$

which is indeed the energy equation. A straight forward way to derive it is to multiply (4.3) by x_t and rearrange.

The second equation is found in the same way by multiplying with x_m:

$$x_m x_{tt} - E'_{mm} = (x_m x_t)_t - x_t x_{mt} - E'_m \cdot x_m\,.$$

Now E is a function of x_m only. Therefore:

$$E_m = E' x_{mm} = (E' x_m)_m - E'_m \cdot x_m\,.$$

Combining these results we obtain:

$$(x_m x_t)_t = (\tfrac{1}{2}x_t^2 - E + E' x_m)_m$$

or

$$(Vv)_t = (\tfrac{1}{2}v^2 - H)_m\,. \tag{4.5}$$

The meaning of this equation is not obvious. We will show in the next section that it is the "translation" of the familiar Bernouilli equation is hydrodynamics from local to material coordinates. In the form (4.5) it can be deduced directly from (4.3) and the identity (2.5):

$$(Vv)_t = Vv_t + vV_t = VYV_m + vv_m\,.$$

Now from (3.4) we have:

$$H_m = H' \cdot V_m = -YVV_m$$

which yields (4.5).

The conserved quantity Vv in (4.5) is essentially the same as the quantity $P - P^*$, considered in the paper by Gilbert and Mollow [1], apart from their restriction to ideal elastic media.

5. Local Coordinates

The local form of the equations of motion can be derived in exactly the same way as in hydrodynamics. This derivation rests on the conservation of mass and momentum. (In a purely mechanical theory like the present the energy equation is not an independent one). We prefer here the alternative procedure of converting the integrated form of the conservation equations as this will shed some light on the meaning of (4.5).

In order to perform this conversion it is only necessary to note that, from (2.3) we have:

$$dm = \frac{dx}{V} = \rho\, dx$$

and that in changing from $(\partial/\partial t)_m$ to $(\partial/\partial t)_x$ we have to correct for the flow through the fixed limits of integration in the second case. For the sake of brevity we will adhere to the questionable practice of writing d/dt and $\partial/\partial t$ in stead of $(\partial/\partial t)_m$ and $(\partial/\partial t)_x$. Therefore:

$$\frac{d}{dt}\int_{m_1}^{m_2} Q\, dm = \frac{d}{dt}\int_{x_1}^{x_2} Q\rho\, dx = \frac{\partial}{\partial t}\int_{x_1}^{x_2} Q\rho\, dx + Q_2\rho_2\frac{\partial x_2}{\partial t} - Q_1\rho_1\frac{\partial x_1}{\partial t} =$$

$$= \frac{\partial}{\partial t}\int_{x_1}^{x_2} Q\rho\, dx + Q\rho v\Big|_1^2 . \tag{5.1}$$

From (4.3) we obtain the material momentum equation:

$$\frac{d}{dt}\int_{m_1}^{m_2} v\, dm = \sigma\Big|_1^2$$

which yields upon conversion by means of (5.1):

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho v\, dx = \sigma - \rho v^2\Big|_1^2 . \tag{5.2}$$

In the same way we obtain from (4.4):

$$\frac{\partial}{\partial t}\rho(\tfrac{1}{2}v^2 + E) = \{-\rho v(\tfrac{1}{2}v^2 + H)\}\Big|_1^2 . \tag{5.3}$$

Finally we convert (4.5) and obtain:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} v\, dx = -(\tfrac{1}{2}v^2 + H)\Big|_1^2 .$$

Now a one-dimensional flow always can be considered as a potential flow:

$$v = \frac{\partial \phi}{\partial x} . \tag{5.4}$$

Therefore (4.5) is equivalent to:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2 + H\Big|_1^2 = 0 \tag{5.5}$$

which is Bernouilli's law.

We still miss a mass equation. This can be derived by writing (2.4) as

$$\frac{d}{dt}\int dm = 0$$

and applying the same procedure. The result is:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho\, dx = -\rho v\Big|_1^2 . \tag{5.6}$$

Treating (2.5) in the same manner simply yields $\partial x/\partial t=0$, which reminds us that (2.5) is essentially trivial.

The equations of motions now result by differentiating (5.2) and (5.6) with respect to x_2:

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2) = \frac{\partial \sigma}{\partial x} \tag{5.7}$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 . \tag{5.8}$$

Subtracting v times the continuity equation (5.8) from the momentum equation (5.6) yields the latter in the familiar Euler form:

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} = \frac{\partial \sigma}{\partial x} . \tag{5.9}$$

Now multiply (5.9) by $\rho^{-1} = V$ and substitute (5.4). Upon integration of the resultant expression we obtain:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2\Big|_1^2 = \int_{x_1}^{x_2} V \frac{\partial \sigma}{\partial x} \cdot dx = V\sigma\Big|_1^2 - \int_1^2 \sigma dV = -H\Big|_1^2$$

which is equation (5.5) again.

6. Linearisation and the Ideal Elastic Bar

When the amplitude of the elastic waves is small, that is when $V/V_0 \approx 1$, linearisation of equations (4.3) or (5.8) and (5.9) obviously is a reasonable approximation. In this case the difference between the operators d/dt, $\partial/\partial s$ and $\partial/\partial t$, $\partial/\partial x$ is of higher order and consequently neglected. The local and material versions of the theory are the same in this approximation. Corresponding approximations are easily performed in the conservation laws. It seems not very useful to go into details. We only mention that the first order parts of (4.4) and (4.5) reduce to (2.5) and (4.3) respectively. The second order parts turn out to be the familiar quadratic conservation laws going with the linear wave equation.

Less trivial is the remark that, for ideal elastic media, the second equation (4.3) remains linear for finite amplitudes. It can be written in the form:

$$x_{tt} = c^2 x_{ss} \tag{6.1}$$

and therefore the complete solution is:

$$x = f(s-ct) + g(s+ct)$$

without further restrictions. The constant phase velocity c clearly is the speed of disturbances with respect of the mass. That is, a disturbance during a time dt sweeps a mass

$$dm = \frac{ds}{V_0} = \frac{c}{V_0} \cdot dt .$$

Waves coming from opposite directions interpenetrate and travel on without any change of shape or loss of time if they are described in terms of mass passed instead of distance covered.

In order to see what becomes of the local equations under these circumstances we remember from gasdynamics that these equations can be simplified by expressing the material quantities in terms of the sound speed a, defined by:

$$a^2 = -\frac{d\sigma}{d\rho} = V^2 \frac{d\sigma}{dV} . \tag{6.2}$$

With (3.5) and (3.8) we obtain:

$$a = \frac{V}{V_0} \cdot c . \tag{6.3}$$

Using this in (5.8) and (5.9) gives, after some rearrangement:

$$\frac{\partial \alpha}{\partial t} + \beta \frac{\partial \alpha}{\partial x} = 0 \tag{6.4}$$

$$\frac{\partial \beta}{\partial t} - \alpha \frac{\partial \beta}{\partial x} = 0 \tag{6.5}$$

where

$$\alpha = a - v$$

$$\beta = a + v .$$

From these equations we see that a is indeed the true wave speed, distance over time, with respect to the medium.

It is seen that (6.4) and (6.5) are non-linear equations. They are however of a rather special type. The coefficients in (6.4), the propagation equation for α, are independent of α. We can, for instance, satisfy (6.5) by putting $\beta = c$. The solution of (6.4) then is $\alpha = f(x - ct)$ as if the problem were linear. The ideal elastic bar behaves like a linear local field as long as waves travelling in one direction only are present. However, when two opposing wave trains meet there is, in the local frame of reference, non-linear distorsion. It is possible to prove that two colliding compressive wave trains of finite extent leave each other spatially undistorted after crossing, both showing however a certain delay in time. A further discussion of the properties of equations (6.4) and (6.5) however is beyond the scope of this paper.

Concluding: The elastic bar behaves approximately as a linear system for small distortions. This applies both to local and material coordinates. The ideal elastic bar behaves exactly as a linear system for all distortions (with the obvious restriction that $V > 0$) in material coordinates. The ideal elastic bar moreover behaves exactly as a linear system in local coordinates for distortions of simple wave type.

7. A Remark on Momentum

Equation (4.3) is to be considered the momentum equation of the bar. Its density v is the momentum per unit mass, its flux density σ is the force in a cross section. Gilbert and Morrow [1] consider an ideal elastic bar, interacting with some external mass M. In this case one finds when the interaction is conservative:

$$\frac{dM}{dt} + \frac{d}{dt} \int v \, dm = 0 \tag{7.1}$$

when it is assumed that interaction and stress vanish at infinity.

Next they show that there exists another density p, of the same dimension as v, such that also:

$$\frac{dM}{dt} + \frac{d}{dt} \int (v + p) \, dm = 0 \tag{7.2}$$

under the same circumstances. Comparing their results with ours one finds that the conservation equation for p essentially is our equation (4.5) divided by V_0. In the paper cited it is proposed to call $v + p$ the tensor momentum of the bar.

In our opinion this is rather confusing and unnecessary. The reason for this is that an infinite number of linear independent quantities p satisfying the same requirements can be constructed.

For the sake of simplicity we will show this for the ideal elastic bar only. Using (6.3), (3.3), (3.8) and (2.1) we can write (4.3) in the form:

$$v_t = c a_s . \tag{7.3}$$

In the same way (2.5) can be transformed into:

$$a_t = cv_s\,. \tag{7.4}$$

This set of equations has, for each number n, two independent conservation laws of degree n. These can be written in the form:

$$\frac{\partial}{\partial t}(a \pm v)^n = \pm c\frac{\partial}{\partial s}(a \pm v)^n\,. \tag{7.5}$$

It is easily shown that (7.5) is an identity for all functions a, v which satisfy (7.3) and (7.4). For $n=1$ we find (7.3) and (7.4), for $n=2$ the energy and Bernouilli equations result. Analogous results can be derived for the local equations (6.4) and (6.5). If we now take $p=c^{(1-n)}(a \pm v)^n$, equation (7.2) will be satisfied without any further restrictions. In our opinion therefore it is advisable to reserve the term "momentum" for quanties whose rate of change is a force.

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